Ideal Variable Spring-Mass-Damper System Analyzed via the Laplace Transform

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Abstract

Utilizing the Laplace Transform method of solving Ordinary Differential Equations (ODEs), an algebraic system with corresponding coefficients can be resolved to find the time-domain function of the original differential system. The goal of this report is to present a second-order, nonhomogeneous, variable coefficient ODE that represents an ideal spring-mass-damper (SMD) system as it evolves under varying the coefficients relating to each term.

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The problem statement begins with the construction of a SMD system using a unit mass, a spring with spring constant $k > 0$, and a damper with damping constant $2d \geq 0$. Initial conditions to the system are zero velocity and zero displacement at $t = 0$. Then, at $t = 0$, a sinusoidal force $f(t) = \sin(\omega t)$, with $\omega \geq 0$, is applied to the mass; resulting in displacement of the mass, $x(t)$, being modelled by the initial value problem:

[1] $$x'' + 2dx' + kx = \sin(\omega t)\,; \quad x(0) = 0, x'(0) = 0$$

With x' and x'' referring likewise to the first and second derivatives of x with respect to t. A general solution to $x(t)$ will then be found using the Laplace Transform of the statement to analyze relationships of the varying constants k, ω, and d.

The Transform and Finding Expressions for Coefficients

The one-sided Laplace Transform is stated as the following integral, with s being the resultant variable in the complex frequency domain:

[2] $$\mathcal{L}\{f(t)\}(s) = \int_0^\infty e^{-st} f(t)\,dt$$

Provided $f(t)$ is of exponential order and piecewise continuous, the Laplace Transform exists for the given function. Computation of exact function antiderivatives is not needed for this problem, as common transforms are available via table lookup. The transformation of the problem statement (expression 1), with the transform of $x(t)$ being denoted $X(s)$, then results in the following expression in the s-domain:

[3] $$s^2X(s) - sf(0) - f'(0) + 2dsX(s) - f(0) + kX(s) = \frac{\omega}{s^2+\omega^2}$$

Solving for $X(s)$:

[4] $$X(s) = \frac{\omega}{(s^2+\omega^2)(s^2+2ds+k)}$$

The RHS of this expression can be simplified via partial fraction decomposition by introducing four additional constants *A*, *B*, *C*, and *D*:

[5] $$\frac{\omega}{(s^2+\omega^2)(s^2+2ds+k)} = \frac{As+B}{(s^2+\omega^2)} + \frac{Cs+D}{(s^2+2ds+k)}$$

Multiplying by the LHS denominator, the linear expression needed to find the four introduced coefficients is produced:

[6] $$\omega = (As+B)(s^2+2ds+k) + (Cs+D)(s^2+\omega^2)$$

After selecting clever values for *s*, such as *s = 0*, and by comparing common powers of *s* on the LHS and RHS, such as s^3, s^2, and s^1, six total linear expressions are produced:

[7] $$s = 0: \quad \omega = kB + \omega^2 D \Rightarrow D = \frac{\omega - kB}{\omega^2}$$

[8] $$s^3: \quad 0 = A + C \Rightarrow C = -A$$

[9] $$s^2: \quad 0 = B + 2dA + D$$

[10] $$s^1: \quad 0 = kA + 2dB \Rightarrow A = \frac{-2dB}{k}$$

The coefficient for B, in terms of constants *k*, *ω*, and *d*, is found after plugging in expressions 7 and 10 into expression 9. Once *B* is found, the other three constants *A*, *C*, and *D*, are then trivial:

[11] $$A = \frac{2d}{k\omega - 4d^2\omega - \frac{k}{\omega}}$$

[12] $$B = -\frac{1}{\omega - \frac{4d^2\omega}{k} - \frac{k^2}{\omega}}$$

[13] $$C = -\frac{2d}{k\omega - 4d^2\omega - \frac{k^2}{\omega}}$$

[14] $$D = \frac{1}{\omega} + \frac{k}{\omega^3 - \frac{4d^2\omega^3}{k} - k\omega}$$

Fortunately, these expressions for *A*, *B*, *C*, and *D* are not needed in their expanded form for the purposes of this paper.

Having found all needed information, the transform is then:

[15] $$X(s) = \frac{As+B}{(s^2+\omega^2)} + \frac{Cs+D}{(s^2+2ds+k)}$$

Substituting $\alpha = d^2 - k$, completing the square in the RHS second denominator, and separating the fractions yields:

[16] $$X(s) = \frac{As+B}{(s^2+\omega^2)} + \frac{C(s+d)-Cd+D}{(s+d)^2-\alpha}$$

And then,

[17] $$X(s) = \frac{As}{s^2+\omega^2} + \frac{B}{s^2+\omega^2} + \frac{C(s+d)}{(s+d)^2-\alpha} - \frac{Cd}{(s+d)^2-\alpha} + \frac{D}{(s+d)^2-\alpha}$$

Finally, computing the inverse Laplace Transform, $x(t)$ is found for $t \geq 0$:

[18] $$x(t) = A\cos(\omega t) + \frac{B}{\omega}\sin(\omega t) + Ce^{-dt}\cosh(t\sqrt{\alpha}) - \frac{Cd+D}{\sqrt{\alpha}}e^{-dt}\sinh(t\sqrt{\alpha})$$

This expression models the displacement of the SMD system with respect to time for all $t \geq 0$ such that $\sqrt{\alpha} \in \mathbb{R}$ and $\alpha > 0$.

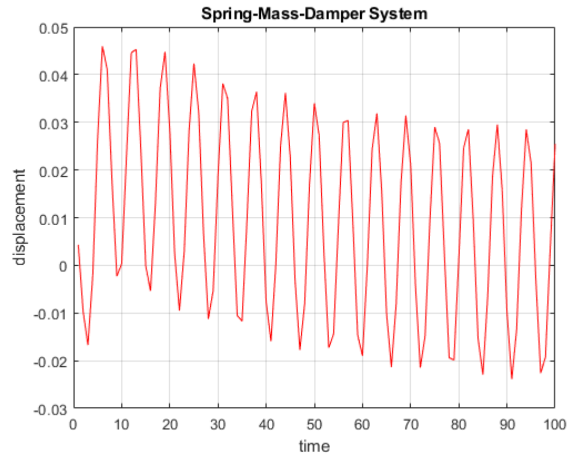
Discussion of Displacement Curve Behavior as Coefficients Vary

By using MATLAB software to model the displacement function (expression 18), allowing the constants k, ω, and d to vary one at a time results in three primary emergent behaviors: exponential growth or decay, high or low relative frequency, and finally amplitude relationships. Following the restrictions outlined in the previous sections: $\alpha > 0$ and real, as well as restricting $\omega \neq 0$ unless the original input force expression $\sin(\omega t)$ is also set to zero, the emergent behavior of the spring mass system can be observed.

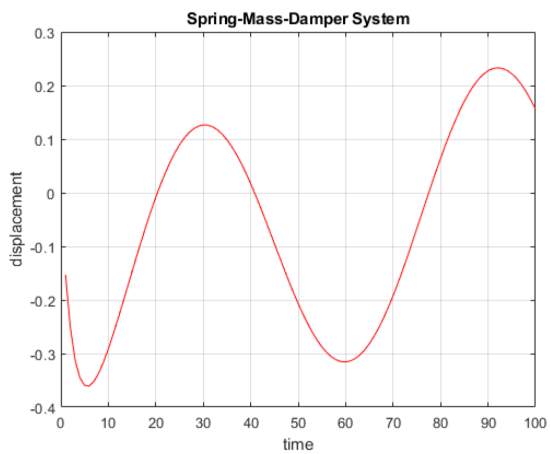
ω Varies

Plotting the displacement of system for ω = *-10*, *1*, and *100*, while setting $k = 1$ and $d = 2$, produces periodic and quasi-periodic output curves. As the magnitude of ω increases, the frequency and average amplitude of the curve also increase.

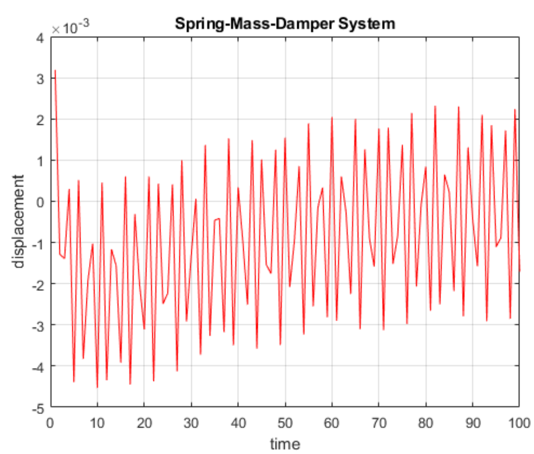
The following figure is ω = -10:



Then for ω = 1:



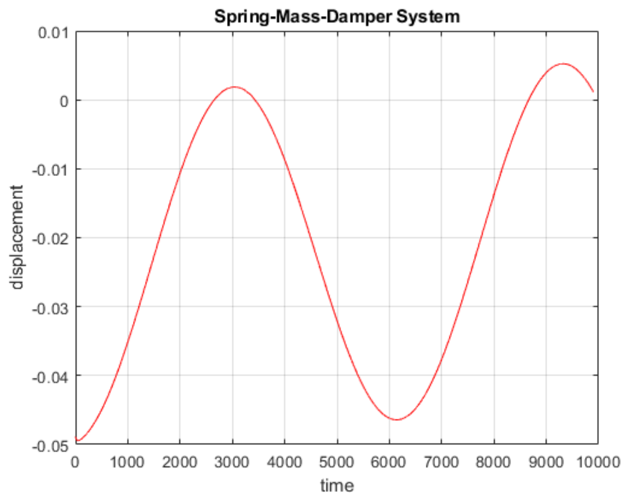
Finally, for ω = 100:



k and *d* Vary

Because α is restricted to being both real and positive, *k* and *d* varying must be handled together. This can be thought of as letting α vary while understanding the underlying constants must vary together or independently such that α meets the given restrictions.

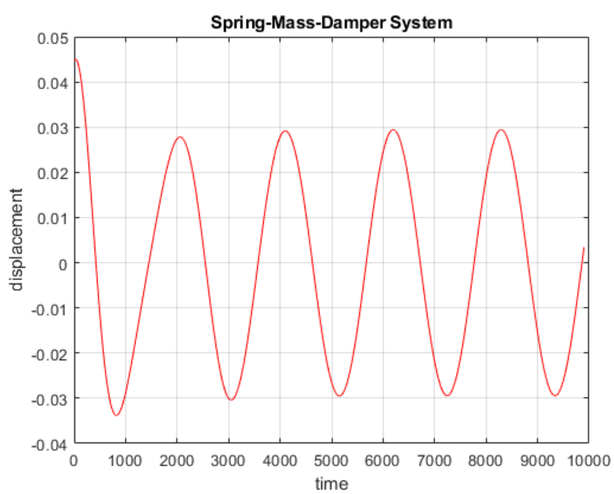
The following figure plots displacement for $d = 20$ and $k = \omega = 1$:



As expected, for large values of damping, along with small frequency of input force (ω) and small spring constant, the overall movement of the SMD system is sluggish and small in frequency and amplitude.

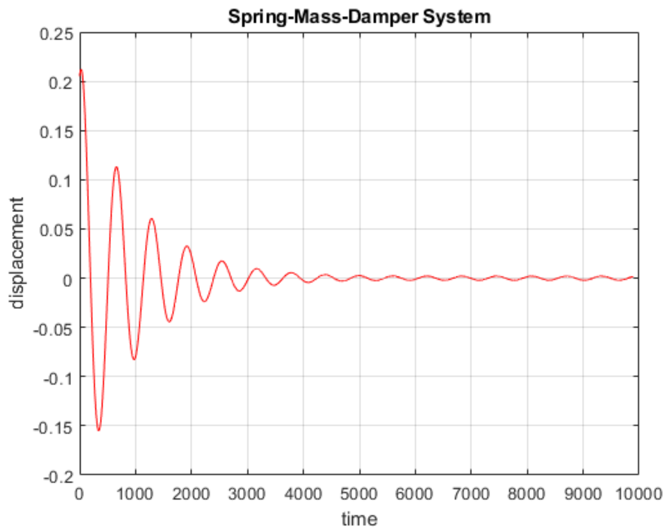
If we handle the Real part of $\sqrt{\alpha}$ alone, then high spring constants with respect to low damping constants can be handled.

The following figure plots displacement for $d = 1$, $k = 30$, and $\omega = 3$:



Note that when k and ω are multiples of each other or "nearly" multiples of each other, rather "smooth" behavior of the SMD system occurs, with either exponentially increasing or decreasing amplitude of displacement over time.

The following figure plots displacement for $d = 1$, $k = 101$, and $\omega = 10$:



Whenever the damping constant d has any significant magnitude, any exponentially decaying or growing behavior is significantly stunted following intuition about SMD systems in the real world.

Conclusion

While rather non-rigorous in detail in describing parameters that predict behavior of the SMD system with variable constant coefficients, intuition about SMD systems is apparent in software modeling using my found expression 18 for displacement over time. Larger damping values result in muffled exponential behavior, input frequency ω is directly correlated to output curve frequency, and α dominating exponential behavior, all correspond to the generalized SMD model behavior with respect to time. Curiously, the handling of the damping and spring constant relationship within α leads to complex domain values which interpreted only with the Real components provides an accurate model. What is the physical significance of this complex frequency term α?